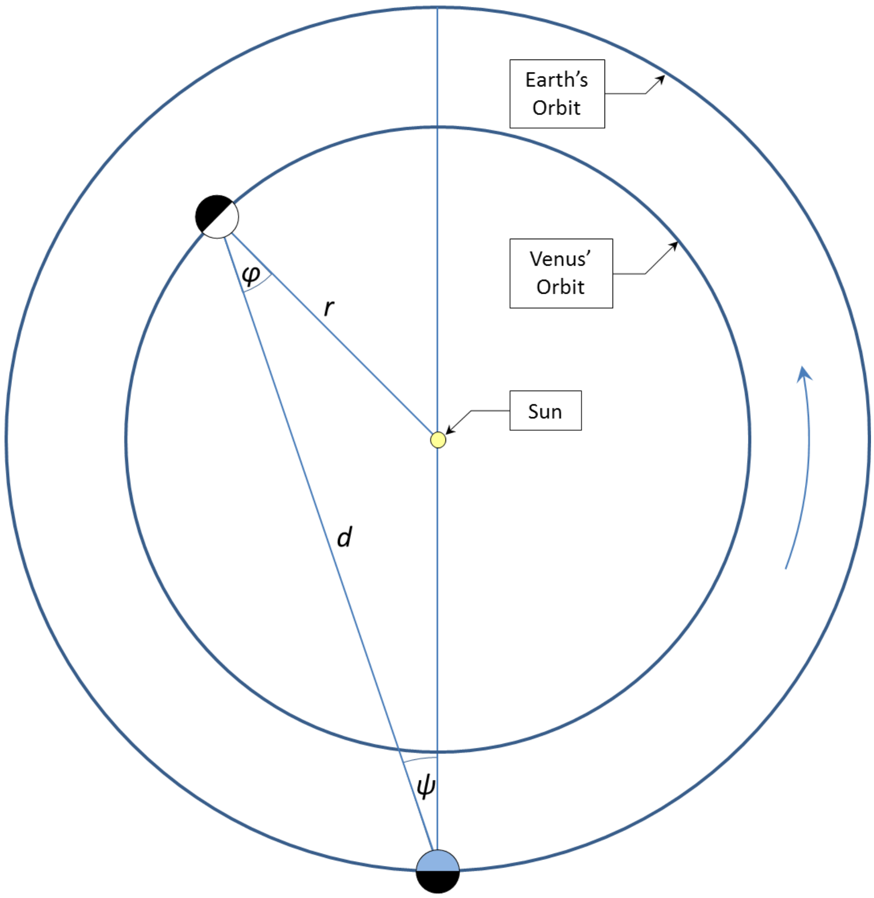
Geometric Maximum Brightness



Let:

p = the phase, the proportion of the area of Venus' disk that is illuminated as viewed from Earth,
φ = the phase angle, the Sun-Venus-Earth angle,
ψ = the elongation of Venus, the Sun-Earth-Venus angle,
r = the radius of Venus' orbit = 0.723 AU,
d = the distance, in AU, from Earth to Venus,
a = the radius of Venus = 6,051.8 km,
A = the surface albedo of Venus' clouds = 0.90 (Bond),
L = the Luminosity of the Sun = 3.846×10^{26} W,
F_0 = the light flux on Venus' surface,

The orbits of both Earth and Venus are considered to be circular and coplanar.

We have the incident flux as:

$$F_0 = \frac{L}{4\pi r^2} \qquad (1)$$

The phase is the illuminated proportion of Venus' disk seen from Earth. From geometry and the law of cosines (see the diagram above) we have:

$$p = \tfrac{1}{2}(1 + \cos\varphi) \qquad (2)$$

$$\cos\varphi = \frac{d^2 + r^2 - 1}{2rd} \qquad (3)$$

$$\therefore p = \frac{(d+r)^2 - 1}{4rd} \qquad (4)$$

The observed flux at Earth is:

$$F_G = \frac{LAp}{4\pi r^2}\frac{\pi a^2}{4\pi d^2} = \frac{LAa^2}{16\pi r^2}\frac{p}{d^2} = \frac{LAa^2}{64\pi r^3}\frac{(d+r)^2 - 1}{d^3} \qquad (5)$$

Setting:

$$k_G = \frac{LAa^2}{64\pi r^3} \qquad (6)$$

we have:

$$F_G = k_G \frac{(d+r)^2 - 1}{d^3} \qquad (7)$$

Differentiating this wrt d gives:

$$\frac{dF_G}{dd} = k_G[-3d^{-4}((d+r)^2 - 1) + 2d^{-3}(d+r)] \qquad (8)$$

For a maximum (or minimum):

$$\frac{dF_G}{dd} = \frac{k_G}{d^4}[2d(d+r) - 3((d+r)^2 - 1)] = 0 \qquad (9)$$

Since $d \neq 0$ we can rearrange this as:

$$d^2 + 4rd + (3r^2 - 3) = 0 \qquad (10)$$

Since $d > 0$ we solve the quadratic equation for the positive solution:

$$d = -2r + \sqrt{r^2 + 3}$$

For r = 0.723 AU this gives d = 0.4309 AU and, substituting in (3), gives φ=117.9° and, using the law of cosines for ψ, gives ψ=39.7°.

Using the angle sum of a triangle we have the Venus-Sun-Earth angle as 180°-117.9°-39.7°=22.4°. For Venus' synodic period of 583.92 days this corresponds to maximum brightness at 22.4° x 583.92 days / 360°= 36.3 days before or after inferior conjunction.

Lambertian Maximum Brightness

Lambert's law for a diffusely reflecting surface gives the power radiated, Δf, per steradian from a small surface element, Δs, in a direction making an angle ε with the surface normal as:

$$\Delta f = \frac{F_0 A}{\pi} \cos i \cos \varepsilon \, \Delta s \tag{11}$$

where symbols defined in the previous section have been re-used and i is the angle of the incident light to the normal.

For the spherical surface of Venus' clouds and expressing the surface element in (11) above in spherical co-ordinates with θ as the polar angle measured away from the axis pointing towards the observer and ω as the azimuthal angle around that axis:

$$\Delta s = a^2 \cos \theta \; \Delta\theta \, \Delta\omega \tag{12}$$

The angles of incidence and reflection are determined by:

$$\cos i = \cos \theta \cos(\omega - \varphi) \tag{13}$$

$$\cos \varepsilon = \cos \theta \cos \omega \tag{14}$$

where φ is the phase angle, the angle between the light source and observer subtended at Venus' centre. Both the light source and observer are assumed to be at infinity to remove the complication of perspective.

Substituting (12), (13) and (14) in (11) and integrating we obtain the total flux from Venus:

$$f = \frac{F_0 A a^2}{\pi} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta \int_{\phi-\pi/2}^{\pi/2} \cos(\omega - \varphi) \cos \omega \, d\omega \tag{15}$$

$$= \frac{2}{3} \frac{F_0 A a^2}{\pi} \{\sin \varphi + (\pi - \varphi) \cos \varphi\} \tag{16}$$

The observed flux at Earth is:

$$F_L = \frac{f}{d^2} = \frac{1}{6} \frac{L A a^2}{\pi^2 r^2} \frac{1}{d^2} \{\sin \varphi + (\pi - \varphi) \cos \varphi\} \tag{17}$$

Setting

$$k_L = \frac{1}{6} \frac{L A a^2}{\pi^2 r^2} \tag{18}$$

we have:

$$F_L = k_L d^{-2} \{\sin \varphi + (\pi - \varphi) \cos \varphi\} \tag{19}$$

Differentiating wrt d gives:

$$
\begin{aligned}
\frac{dF_L}{dd} &= k_L\left[-2d^{-3}\{\sin\varphi + (\pi-\varphi)\cos\varphi\} + d^{-2}\left\{\cos\varphi\frac{d\varphi}{dd} - (\pi-\varphi)\sin\varphi\frac{d\varphi}{dd} - \cos\varphi\frac{d\varphi}{dd}\right\}\right] \\
&= k_L\left[-2d^{-3}\sin\varphi - 2d^{-3}(\pi-\varphi)\cos\varphi - d^{-2}(\pi-\varphi)\sin\varphi\frac{d\varphi}{dd}\right]
\end{aligned} \tag{20}
$$

Differentiating (3) wrt d gives:

$$
-\sin\varphi\frac{d\varphi}{dd} = \frac{1}{2r}\left[-d^{-2}(r^2+d^2-1)+2\right] = \frac{1}{2rd^2}(d^2-r^2+1) \tag{21}
$$

Substituting (3) and (21) into (20) and for a maximum or minimum gives:

$$
\frac{dF_L}{dd} = \frac{k_L}{2rd^4}\left[(\pi-\varphi)(d^2-r^2+1) - 4rd\{\sin\varphi + (\pi-\varphi)\cos\varphi\}\right] = 0 \tag{22}
$$

Noting again that $d \neq 0$ rearrangement gives:

$$
(\pi-\varphi)(d^2+3r^2-3) + 4rd\sin\varphi = 0 \tag{23}
$$

Equation (23) cannot be solved in a closed form so numerical methods must be used to determine its solution.

The solution gives d = 0.5401 AU, φ=103.74° and, using the law of cosines for ψ, gives ψ=44.61°.

Using the angle sum of a triangle we have the Venus-Sun-Earth angle as 180°-103.74°-44.61°=31.65°. For Venus' synodic period of 583.92 days this corresponds to maximum brightness at 31.65° x 583.92 days / 360°= 51.34 days before or after inferior conjunction.