[Physics FAQ] - [Copyright]

Updated by Don Koks 2006.
Fuel numbers added by Don Koks 2004.
Updated by Phil Gibbs 1998.
Thanks to Bill Woods for correcting the fuel equation.
Original by Philip Gibbs 1996.

The Relativistic Rocket

The theory of relativity sets a severe limit to our ability to explore the galaxy in space ships. As an object approaches the speed of light, more and more energy is needed to accelerate it further. To reach the speed of light an infinite amount of energy would be required. It seems that the speed of light is an absolute barrier which cannot be reached or surpassed by massive objects (see relativity FAQ article on faster than light travel). Given that the galaxy is about 100,000 light years across there seems little hope for us to get very far in galactic terms unless we can overcome our own mortality.

Science fiction writers can make use of worm holes or warp drives to overcome this restriction, but it is not clear that such things can ever be made to work in reality. Another way to get around the problem may be to use the relativistic effects of time dilation and length contraction to cover large distances within a reasonable time span for those aboard a space ship. If a rocket accelerates at 1*g* (9.81 m/s^2) the crew will experience the equivalent of a gravitational field with the same strength as that on Earth. If this could be maintained for long enough they would eventually receive the benefits of the relativistic effects which improve the effective rate of travel.

What then, are the appropriate equations for the relativistic rocket?

First of all we need to be clear what we mean by continuous acceleration at 1*g*. The acceleration of the rocket must be measured at any given instant in a non-accelerating frame of reference travelling at the same instantaneous speed as the rocket (see relativity FAQ on accelerating clocks). This acceleration will be denoted by *a*. The proper time as measured by the crew of the rocket (i.e. how much they age) will be denoted by *T*, and the time as measured in the non-accelerating frame of reference in which they started (e.g. Earth) will be denoted by *t*. We assume that the stars are essentially at rest in this frame. The distance covered as measured in this frame of reference will be denoted by *d* and the final speed *v*. The time dilation or length contraction factor at any instant is the gamma factor γ.

The relativistic equations for a rocket with constant positive acceleration $a > 0$ are the following. First, define the hyperbolic trigonometric functions sh, ch, and th (also known as sinh, cosh, and tanh):

$$\text{sh } x = (e^x - e^{-x})/2$$

$$\text{ch } x = (e^x + e^{-x})/2$$

$$\text{th } x = \text{sh } x/\text{ch } x$$

Using these, the rocket equations are

$$t = (c/a)\ \text{sh}(aT/c) = \text{sqrt}[(d/c)^2 + 2d/a]$$

```
d = (c²/a) [ch(aT/c) - 1] =  (c²/a) (sqrt[1 + (at/c)²] - 1)

v = c th(aT/c) = at / sqrt[1 + (at/c)²]

T = (c/a) sh⁻¹(at/c) = (c/a) ch⁻¹ [ad/c² + 1]

γ = ch(aT/c) = sqrt[1 + (at/c)²] = ad/c² + 1
```

These equations are valid in any consistent system of units such as seconds for time, metres for distance, metres per second for speeds and metres per second squared for accelerations. In these units $c = 3 \times 10^8$ m/s (approx). To do some example calculations it is easier to use units of years for time and light years for distance. Then c = 1 lyr/yr and g = 1.03 lyr/yr^2. Here are some typical answers for a = 1g.

T	t	d	v	γ
1 year	1.19 yrs	0.56 lyrs	0.77c	1.58
2	3.75	2.90	0.97	3.99
5	83.7	82.7	0.99993	86.2
8	1,840	1,839	0.9999998	1,895
12	113,243	113,242	0.99999999996	116,641

So in theory you can travel across the galaxy in just 12 years of your own time. If you want to arrive at your destination and stop then you will have to turn your rocket around half way and decelerate at 1g. In that case it will take nearly twice as long in terms of proper time T for the longer journeys; the Earth time t will be only a little longer, since in both cases the rocket is spending most of its time at a speed near that of light. (We can still use the above equations to work this out, since although the acceleration is now negative, we can "run the film backwards" to reason that they still must apply.)

Here are some of the times you will age when journeying to a few well known space marks, arriving at low speed:

4.3 ly	nearest star	3.6 years
27 ly	Vega	6.6 years
30,000 ly	Center of our galaxy	20 years
2,000,000 ly	Andromeda galaxy	28 years
n ly	anywhere, but see next paragraph	1.94 arccosh (n/1.94 + 1) years

For distances bigger than about a thousand million light years, the formulas given here are inadequate because the universe is expanding. General Relativity would have to be used to work out those cases.

If you wish to pass by a distant star and return to Earth, but you don't need to stop there, then a looping route is better than a straight-out-and-back route. A good course is to head out at constant acceleration in a direction at about 45 degrees to your destination. At the appropriate point you start a long arc such that the centrifugal acceleration you experience is also equivalent to earth gravity. After 3/4 of a circle you decelerate in a straight line until you arrive home.

Below the rocket, something strange is happening...

In the rocket, you can make measurements of the world around you. One thing you might do is ask how the distance to an interesting star you are headed towards changes with T, the time on your clock. At blast-off ($t=T=0$) the rocket is at rest, so this distance initially equals the distance D to the star in the non-accelerating frame. But once you are moving, however you choose to measure this distance, it will be reduced by your current distance *d* travelled in the non-accelerating frame, as well as the whole lot contracted by a factor of γ, your Lorentz factor at time *T*. Eventually you will pass the star and it will recede behind you. The distance you measure to it at time *T* is

$$(D - d)/\gamma = (D + c^2/a)/\mathrm{ch}(aT/c) - c^2/a$$

A plot of this distance as a function of *T* shows that, as expected, it starts at *D*, then reduces to zero as you pass the star. Then it becomes negative as the star moves behind you. As *T* goes to infinity, the distance asymptotes to a value of $-c^2/a$. That means that everything in the universe is falling "below" the rocket, but never receding any farther than a distance of $-c^2/a$ as measured by you. It all piles up just short of this distance, asymptoting to a plane called a *horizon*. You see this horizon actually form as the rocket accelerates, because there comes a time when no signal emitted from "below" the horizon can ever reach you. Everything falls toward that plane, and as it does so it begins to redden, due to the increasing red shift of its light, because you are accelerating. Finally it fades out of visibility. In fact, as anything gets closer to the horizon, it ages more and more slowly; time comes to a complete halt there. The horizon is a dark plane that appears to be swallowing everything in the universe! But of course, nothing strange is noticed by the non-accelerating Earth observers. There is no horizon anywhere for them.

And inside the rocket, something strange is also happening...

Whereas time slows to a stop a certain distance below the rocket, it speeds up "above" the rocket (that is, in the direction in which it's travelling). This effect could, in principle, be measured inside the rocket too: a clock attached to the rocket's ceiling (i.e. furthest from the motor) ages faster than a clock attached to its floor.

For a standard-sized rocket with a survivable acceleration, this difference in how fast things age within its cabin is very small. Even so, it tells us something fundamental about gravity, via Einstein's *Equivalence Principle*. Einstein postulated that any experiment done in a real gravitational field, provided that experiment has a fairly small spatial extent and doesn't take very long, will give a result indistinguishable from the same experiment done in an accelerating rocket. So the idea that the rocket's ceiling ages faster than its floor (and that includes the ageing of any bugs sitting on these) transfers to gravity: the ceiling of the room in which you now sit is ageing faster than its floor; and your head is ageing faster than your feet. Earth's rotation complicates this effect, but doesn't alter it completely.

This difference in ageings on Earth has been verified experimentally. In fact, it was absolutely necessary to take into account when the GPS satellite system was assembled.

How much fuel is needed?

Sadly there are a few technical difficulties you will have to overcome before you can head off into space. One is to create your propulsion system and generate the fuel. The most efficient theoretical way to propel the rocket is to use a "photon drive". It would convert mass to photons or other massless particles which shoot out the back. Perhaps this may even be technically feasible if we ever produce an antimatter-driven "graser" (gamma ray laser).

Remember that energy is equivalent to mass, so provided mass can be converted to 100% radiation by means of matter-antimatter annihilation, we just want to find the mass *M* of the fuel required to accelerate the payload m. The answer is most easily worked out by conservation of energy and momentum.

First: conservation of energy

The total energy before blast-off is (in the Earth frame)

$$E_{initial} = (M+m)c^2$$

At the end of the trip the fuel has all been converted to light with energy E_L, so the total energy is now

$$E_{final} = \gamma mc^2 + E_L$$

By conservation of energy these must be equal, so here is our first conservation equation:

$$(M+m)c^2 = \gamma mc^2 + E_L \quad \text{........ (1)}$$

Second: conservation of momentum

The total momentum before blast-off is zero in the Earth frame.

$$p_{initial} = 0$$

At the trip's end the fuel has all been converted to light with momentum of magnitude E_L/c, but in the opposite direction to the rocket. So the final momentum is

$$p_{final} = \gamma\ mv - E_L/c$$

By conservation of momentum these must be equal, so our second conservation equation is:

$$0 = \gamma\ mv - E_L/c \quad \text{........ (2)}$$

Eliminating E_L from equations (1) and (2) gives

$$(M+m)c^2 - \gamma mc^2 = \gamma mvc$$

so that the fuel:payload ratio is

$$M/m = \gamma(1 + v/c) - 1$$

This equation is true irrespective of how the ship accelerates to velocity v, but if it accelerates at constant rate a then

$$\begin{aligned} M/m &= \gamma(1 + v/c) - 1 \\ &= \cosh(aT/c)[\ 1 + \tanh(aT/c)\ ] - 1 \\ &= \exp(aT/c) - 1 \end{aligned}$$

How much fuel is this? The next chart shows the amount of fuel needed (M) for every kilogramme of payload (m=1 kg).

d	Not stopping, sailing past:	M
4.3 ly	Nearest star	10 kg
27 ly	Vega	57 kg
30,000 ly	Center of our galaxy	62 tonnes
2,000,000 ly	Andromeda galaxy	4,100 tonnes

This is a lot of fuel—and remember, we are using a motor that is 100% efficient!

What if we prefer to stop at the destination? We accelerate to the half way point at 1*g* and then immediately switch the direction of our rocket so that we now decelerate at 1*g* for the rest of second half of the trip. The calculations here are just a little more involved since the trip is now in two distinct halves (and the equations at the top assume a positive acceleration only). Even so, the answer turns out to have exactly the same form: M/m = exp(aT/c) - 1, except that the proper time T is now almost twice as large as for the non-stop case, since the slowing-down rocket is losing the ageing benefits of relativistic speed. This dramatically increases the amount of fuel needed:

d	Stopping at:	M
4.3 ly	Nearest star	38 kg
27 ly	Vega	886 kg
30,000 ly	Center of our galaxy	955,000 tonnes
2,000,000 ly	Andromeda galaxy	4.2 thousand million tonnes

Compare these numbers to the previous case: they are hugely different! Why should that be? Let's take the case of Laurel and Hardy, two astronauts travelling to Vega. Laurel speeds past without stopping, and so only needs 57 kg of fuel for every 1 kg of payload. Hardy wishes to stop at Vega, and so needs 886 kg of fuel for every 1 kg of payload. Laurel takes almost 28 Earth years for the trip, while Hardy takes 29 Earth years. (They both take roughly the same amount of Earth time because they are both travelling close to speed c for most of the journey.) They travel neck-and-neck until they've both gone half way to Vega, at which point Hardy begins to decelerate.

It's useful to think of the problem in terms of relativistic mass, since this is what each rocket motor "feels" as it strives to maintain a 1*g* acceleration or deceleration. The relativistic mass of each traveller's rocket is continually decreasing throughout their trip (since it's being converted to exhaust energy). It turns out that at the half way point, Laurel's total relativistic mass (for fuel plus payload) is about 28*m*, and from here until the trip's end, this relativistic mass only decreases by a tiny amount, so that Laurel's rocket needs to do very little work. So at the halfway point his fuel:payload ratio turns out to be about 1.

For Hardy, things are different. He needs to decrease his relativistic mass to *m* at the end where he is to stop. If his rocket's total relativistic mass at the halfway point were the same as Laurel's (28*m*), with a fuel:payload ratio of 1, Hardy would need to decrease the relativistic mass all the way down to *m* at the end, which would require more fuel than Laurel had needed. But Hardy wouldn't have this much fuel on board—unless he ensures that he takes it with him initially. This extra fuel that he must carry from the start becomes more payload (a lot more), which needs yet more fuel again to carry that. So suddenly his fuel requirement has increased enormously. It turns out that at the half way point, all this extra fuel gives Hardy's rocket a total relativistic mass of about 442*m*, and his fuel:payload ratio turns out to be about 29.

Another way of looking at this odd situation is that both travellers know that they must take fuel on board initially to push them at 1*g* for the total trip time. They don't care about what's happening outside. In that case, Laurel travels for 28 Earth years but ages just 3.9 years, while Hardy travels for 29 Earth years but ages 6.6 years. So Hardy has had to sit at his controls and burn his rocket for almost twice as long as Laurel, and that has required more fuel, with even more fuel required because of the fuel-becomes-payload situation that we mentioned above.

This fuel-becomes-payload problem is well known in the space programme: part of the reason the Saturn V moon rocket was so big was because it needed yet more fuel just to carry the fuel it was already carrying.

Other fuel options

Well, this is probably all just too much fuel to contemplate. There are a limited number of solutions that don't violate energy-momentum conservation or require hypothetical entities such as tachyons or worm holes.

It may be possible to scoop up hydrogen as the rocket goes through space, using fusion to drive the rocket. This would have big benefits because the fuel would not have to be carried along from the start. Another possibility would be to push the rocket away using an Earth-bound grazer directed onto the back of the rocket. There are a few extra technical difficulties but expect NASA to start looking at the possibilities soon :-).

You might also consider using a large rotating black hole as a gravitational catapult but it would have to be *very* big to avoid the rocket being torn apart by tidal forces or spun at high angular velocity. If there is a black hole at the centre of the Milky Way, as some astronomers think, then perhaps if you can get that far, you can use this effect to shoot you off to the next galaxy.

One major problem you would have to solve is the need for shielding. As you approach the speed of light you will be heading into an increasingly energetic and intense bombardment of cosmic rays and other particles. After only a few years of 1*g* acceleration even the cosmic background radiation is Doppler shifted into a lethal heat bath hot enough to melt all known materials.

For the derivation of the rocket equations see "Gravitation" by Misner, Thorne and Wheeler, Section 6.2.